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**LETTER TO THE EDITOR**

# The number of incipient spanning clusters in two-dimensional percolation

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**Abstract.** Using methods of conformal field theory, we conjecture an exact form for the probability that $n$ distinct clusters span a large rectangle or open cylinder of aspect ratio $k$, in the limit when $k$ is large.

The study of the structure of large clusters at the percolation threshold continues to pose interesting problems whose solution sheds light on the nature of the critical state in general. Recently, some attention has been paid to *incipient spanning clusters* (ISCs). These are clusters which connect two disjoint segments of the boundary of a macroscopically large region. Langlands *et al* [1] conjectured that the probability that at least one such cluster exists (that is, that the segments are connected) is invariant under conformal transformations. This statement was placed in the context of conformal field theory in [2], where an explicit formula was given for this crossing probability. (For a review of the status of conformal invariance in critical percolation, see [3].)

More recently, Aizenman [4] has considered, among other things, the probability that there exist $n$ distinct ISCs connecting the two segments†. In the case of a rectangular region, $[0, kL] \times [0, L]$, he has proved that the probability $P(n, k, L)$ that the strip is traversed (in the direction of length $kL$) by $n$ independent clusters satisfies the bounds

$$A\,\mathrm{e}^{-\alpha n^2 k} \leqslant P(n, k, L) \leqslant \mathrm{e}^{-\alpha' n^2 k} \tag{1}$$

where $\alpha$ and $\alpha'$ are (different) constants. Note that, on the basis of scale invariance at the critical point, $P(n, k, L)$ is expected to have a finite limit as $L \to \infty$.

In this letter we extend the arguments of [2] to determine the exact behaviour of the scaling limit of $P$ for large $k$, namely that

$$\lim_{L\to\infty} \ln P(n, k, L) \sim -\tfrac{2}{3}\pi n(n - \tfrac{1}{2})k \tag{2}$$

as $k \to \infty$ for any $n$.

An analogous problem may be posed on a open-ended cylinder of circumference $L$ and length $kL$. In this case we find, for $n \geqslant 2$,

$$\lim_{L\to\infty} \ln P(n, k, L) \sim -\tfrac{2}{3}\pi (n^2 - \tfrac{1}{4})k. \tag{3}$$

† One may distinguish the probability of exactly $n$ ISCs from that of at least $n$. However, in the limits considered in this note, these will turn out to be asymptotically the same.

 

In passing, we note that, as observed by Aizenman [4], the form of the result in (2) and (3) is not surprising, even though it contradicts what appears to have been a former consensus that only one such cluster should exist which has only recently been challenged and corrected by numerical evidence [5, 6]. Indeed, if one imagines dividing the rectangle into two equal rectangles each of size $[0, kL] \times [0, L/2]$, and assumes that the dominant event will be that of approximately $n/2$ clusters spanning each half, then, up to prefactors,

$$P(n, k, L) \sim P((n/2), 2k, L/2)^2. \tag{4}$$

Together with the expected exponential dependence on $kL$ at fixed $L$, this leads to the above form. A similar argument may then be made for the cylindrical geometry.

Our argument for the exact coefficients in (2) and (3) is based on the well known mapping of bond percolation to the $q \to 1$ limit of the $q$-state Potts model, and the understanding of the critical theory of this model through conformal field theory. In the Potts model, spins $s_i$ are placed at the sites $i$ of a lattice, each taking one of $q$ possible states. The partition function has the form $\mathrm{Tr}\prod_{ij}(1 - p + p\delta_{s_i s_j})$, where the product is over all links of the lattice. This may be expanded in powers of $p/(1 - p)$ so that each term corresponds to a particular realization of bond percolation, weighted by a factor of $q$ for each connected cluster. The limit $q \to 1$ then weights these as in percolation, but, as will be seen, it is also often helpful to consider first the case of more general $q$.

In the rectangular geometry described above, it is useful to express things in terms of the transfer matrix $\mathrm{e}^{-\hat{H}(L)}$ for a strip of width $L$. The partition function for the Potts model with particular boundary conditions at either end of a strip of finite length $kL$ then has the form

$$\langle A|\mathrm{e}^{-kL\hat{H}(L)}|A\rangle \tag{5}$$

where $|A\rangle$ is a *boundary state* corresponding to the boundary conditions chosen. The symmetry of the Potts model ensures that the degenerate subspaces of eigenstates of $\hat{H}(L)$ may be chosen to transform according to irreducible representations of the permutation group $S_q$ of $q$ objects. Conformal field theory also asserts that, in the scaling limit, the states in the low-lying spectrum of $\hat{H}(L)$ transform according to highest weight representations of a Virasoro algebra, and their corresponding eigenvalues have the form $\pi(x + \text{integer})/L$, where $x$ is the highest weight. Thus (5) may also be written

$$\sum_R \mathrm{e}^{-\pi x_R k} \sum_N \langle A|N\rangle\langle N|A\rangle\, \mathrm{e}^{-\pi N k} \tag{6}$$

where the first sum is over highest weight representations $R$, and the second over the states $|N\rangle$ is each representation. (This notation is a little corrupt because there are in general many states at level $N$.)

The simplest non-trivial irreducible representation of $S_q$ has dimension $q - 1$, corresponding to a vector $(\varphi_1, \ldots, \varphi_q)$ with $\sum_{a=1}^{q} \varphi_a = 0$. An example is the Potts order parameter $\varphi_a = \delta_{s_i,a} - q^{-1}$. Out of this, other representations may be built by taking direct products. For example symmetric tensors $\varphi_{ab}$ with $a \neq b$ and $\sum_a \varphi_{ab} = 0$ give a representation of dimension $(q - 1)(q - 2)/2$. In general, we may construct tensors $\varphi_{ab\ldots}$ with $n$ components, none of whose indices are equal. Let us denote by $R_n$ the Virasoro representation which also carries this representation of $S_q$ and which has the smallest weight $x_R$. Denote this weight by $x_n$.

Suppose now that we are interested in those configurations in which at least $n$ distinct ISCs connect the two ends of the strip. In that case it is possible to colour these clusters with $n$ different colours of the Potts model, and therefore the states which propagate along the strip must carry at least $n$ different colours. If there are fewer than $n$ ISCs, it is not possible to make such an assignment. In the limit of large $k$, then the partition sum in (5) will be dominated by those state(s) transforming according to representations $R_{n'}$ with $n' \geqslant n$. As we shall argue, the highest weights $x_{n'}$ are monotonically increasing in $n'$. Thus the states with $n' = n$ dominate the sum.

What is the value of $x_n$? For $n = 1$ the answer is known, since it corresponds to the scaling dimension of the Potts order parameter near the boundary of a semi-infinite system. It was conjectured in [7] that this corresponds to the operator $(1, 3)$ in the Kac classification [8, 9], giving, for general $q$, $x_1 = (m-1)/(m+1)$, where $q = 4\cos^2(\pi/(m+1))$, and $x_1 = \frac{1}{3}$ for $q = 1$. This conjecture agrees with the known exact result for $q = 2$ and numerical work for $q = 3$ and $q = 1$. Its correctness is also born out by the numerical success of the crossing formula of [2]. We now further conjecture that the representations $R_n$ correspond to $(1, 2n+1)$ in the Kac classification. This is based on the fusion rules for these representations. Observe that the composition law for the $S_q$ representations under consideration is isomorphic to that for addition of spin $n$ in $SU(2)$, which in turn is the same as the fusion rules [8] for the Kac representations $(1, 2n+1)$ in conformal field theory (in non-minimal models corresponding to generic values of $q$). Thus, for example, insertion of two order parameters $\varphi_a$ and $\varphi_b$ near the end of the strip will, in general, give rise to propagating states corresponding to the tensor representation $\varphi_{ab}$ (when $a \neq b$), the vector representation (when $a = b$) and the identity representation. Since these last two correspond to $(1, 3)$ and $(1, 1)$, respectively, we may identify the first with $(1, 5)$. This argument may be generalized straightforwardly to higher values of $n$. Then, according to the Kac formula [8], the highest weight of the $(1, 2n+1)$ representation is $x_n = n(mn-1)/(m+1)$, or $x_n = n(2n-1)/3$ for $q = 1$. This, combined with (6), gives the first result (2), valid as $k \to \infty$ at fixed $n$.

The large $n$ behaviour of (2) may also be derived directly from Coulomb gas arguments [10]. In this approach, the configurations of the critical cluster model are mapped onto those of densely packed loops on the surrounding lattice. Each loop carries a factor $q^{1/2}$, which may be traded for local weights by considering each loop as corresponding to two oriented loops with vertex weights $\mathrm{e}^{\pm \mathrm{i}\chi}$ according to whether they turn to the right or left at a given site, and setting $q^{1/2} = 2\cos 4\chi$. These loop configurations are then mapped onto those of a local height model on the dual lattice, with heights $\phi(r) \in (\pi/2)\mathbb{Z}$, and the rule that the height difference between neighbouring dual sites is $\pm\pi/2$ according to the orientation of the corresponding dual bond. This in turn is supposed to renormalize onto a Gaussian model with reduced Hamiltonian $(g/4\pi)\int(\nabla\phi)^2\,\mathrm{d}^2r$, where $g = 2(2 - 8\chi/\pi)$, and $2 \leqslant g \leqslant 4$. Free boundary conditions on the original Potts model correspond to Dirichlet conditions $\phi =$ constant in the height model.

In the strip geometry, the total charge, that is the number of left-oriented minus right-oriented loops, is conserved along the strip. Consider the configurations where this charge is $2n$. These correspond to cluster configurations where at least $n$ distinct clusters traverse the strip, as illustrated in figure 1. In the height model this means that the difference in the heights between the upper and lower edges is fixed to be $2n(\pi/2) = n\pi$. Neglecting fluctuations, the energy of such a configuration is simply $(g/4\pi)(n\pi/L)^2 \cdot kL \cdot L = (g\pi n^2/4)k$. Inserting the $q = 1$ value $g = \frac{8}{3}$ gives the leading term in the result (2) for $\ln P$. This calculation works only for large $n$ because (a) the mapping to the Gaussian model is valid only in the bulk and not close to the boundary, and only for large $n$ are most of the loops far from the boundary; and (b) because the fluctuations are expected to give an $O(1)$ contribution.

A similar argument, in this case yielding the exact result at large $k$, may be applied to the cylindrical geometry, where there are periodic boundary conditions around the strip.

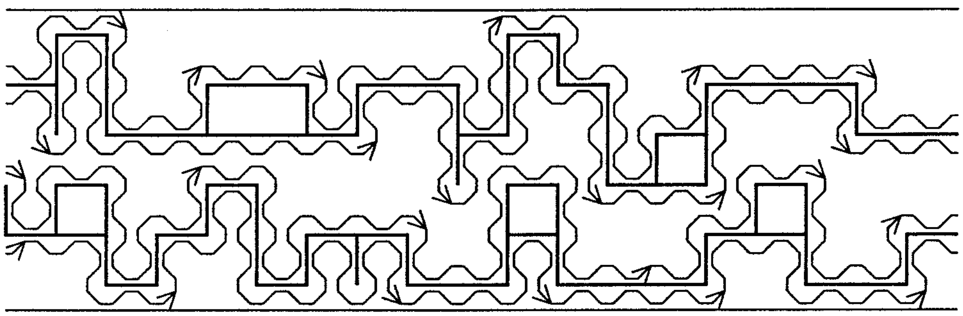

**Figure 1.** The configuration in which two clusters span the strip. The hulls of these correspond to a loop configuration with charge 4. Other possible non-spanning clusters are not shown.

Once again, the loop configurations with charge $2n$ correspond to at least $n$ clusters connecting the ends of the cylinders. Writing $\phi = n\pi v/L + \phi'$, where $v$ is the coordinate around the cylinder and $\phi'$ satisfies periodic boundary conditions, the energy functional is $(g\pi n^2/4)k + (g/4\pi)\int(\nabla\phi')^2\,\mathrm{d}^2r$, where the first term is identical to that for the strip with free boundaries. The integral over the fluctuating part then gives a contribution [11] $(\pi c_{\mathrm{G}}/6)k$ to $\ln P$, where $c_{\mathrm{G}} = 1$ is the central charge of the free scalar field $\phi'$. Putting these together and setting $g = \frac{8}{3}$ then gives the result in (3). Note that this is correct only at $q = 1$: in general it should be normalized by the partition function. The finite-size corrections to this have the above form, with $c = 0$ at $q = 1$. This comes about because the Gaussian result $c_{\mathrm{G}} = 1$ is reduced by the effects of loops which can wind around the cylinder [12]. These are forbidden when other loops already extend along the cylinder, so that no similar reduction occurs in this case.

In general, the behaviour in (3) should be of the form $2\pi x_n^{(b)}k$, where $x_n^{(b)}$ is a *bulk* exponent [13]. In fact, these exponents are the so-called *multi-hull* scaling dimensions discussed by Saleur and Duplantier [14]. In the plane, these determine the power-law decay $\sim|r_1 - r_2|^{-2x_n^{(b)}}$ of the probability that two points $r_1$ and $r_2$ lie in the vicinity of the external boundaries, or hulls, of $n$ distinct clusters. These are computed in the loop gas in terms of configurations with $2n$ oriented lines running from $r_1$ to $r_2$, which is precisely what we have argued above determines $\ln P$ for large $k$. Our result in (3) agrees with that of [14] for these exponents.

As indicated, (3) does not hold for $n = 1$. This is because a single cluster which connects the ends of the cylinder is also allowed to wrap around it: this is clearly not allowed for $n \geqslant 2$. For $n = 1$ the equivalence to the hull exponents no longer holds. Instead, we expect for large $k$ that $P(1, k, L)$ is asymptotically equal to the probability that the two ends are connected (by any number of clusters) and it should behave as $\exp(-2\pi\tilde{x}k)$, where $\tilde{x} = \frac{5}{48}$ is the usual magnetic scaling dimension of the $q = 1$ Potts model, which gives the probability $\sim|r_1 - r_2|^{-2\tilde{x}}$ that points $r_1$ and $r_2$ in the plane are connected. Thus for $n = 1$ on the cylinder, (3) is replaced by

$$\ln P(1, k, L) \sim -(5\pi/24)k. \tag{7}$$

However, the result in (3) with $n = 1$ does have a physical meaning: it is the asymptotic probability that the two ends of the cylinder are connected by a cluster which does not also wrap around the cylinder. As expected, this is much smaller than the unrestricted probability.

Finally, we discuss whether it is possible to compute $P(n,k,L)$ for non-asymptotic values of $k$ and $n$, in the scaling limit $L\to\infty$. This corresponds to a generalization of the calculation of [2], and first involves identifying suitable boundary conditions corresponding to the states $|A\rangle$. It is not difficult to see that these states should be suitable linear combinations of states corresponding to boundary conditions in which each Potts spin is constrained to lie in a subset of $n$ states out of the possible $q$. Let us denote the boundary state in which each spin is constrained to take the values $a$ or $b$ or ... (where all the $a, b, \ldots$ are different) by $|ab\ldots\rangle$. Then $P(n,k,L)\propto\langle A_n|\mathrm{e}^{-kL\hat{H}(L)}|A_n\rangle$, where, for example,

$$|A_1\rangle = |a\rangle - |b\rangle \tag{8}$$

$$|A_2\rangle = |ab\rangle + |cd\rangle - |ac\rangle - |bd\rangle \tag{9}$$

and so on. It may be seen that that these states do indeed transform according to the advertized representations of $S_q$, and so will couple precisely to the representations $R_n$, and this will be the dominant coupling in the limit $k\to\infty$. However, in order to determine the dependence of $P(n,k,L)$ for finite $k$, in analogy with the argument of [2], it is necessary to determine the four-point function of boundary condition changing operators which connect the above boundary conditions to the free boundary conditions along the other edges. Unlike the case of [2] it does not appear that these operators, for $n>1$, correspond to simple Kac representations. However, it may still be possible to conjecture a suitable differential equation or an integral representation for this function, as was done recently by Watts [15] in the case of the probability of a simultaneous left-right and up-down crossing of the rectangle. A simpler case to consider might be that of clusters which span a cylinder of finite length. It is known that in this case the appropriate matrix elements may be expressed as a linear combination of Virasoro characters [16].

We note that the simple argument given above in (4) (and the rigorous arguments of Aizenman [4]) provide a simple physical reason why the scaling dimensions of composite operators such as those discussed should increase like $n^2$ in two dimensions. The generalization of Aizenman's argument to $d$ dimensions suggests that the rate of increase of $-\ln P(n,k,L)$ is like $n^{d/(d-1)}$. However, for $d>2$ this quantity is no longer related to scaling dimensions by conformal invariance.

Our conjecture that the relevant scaling dimensions in (2) are the $(1,2n+1)$ operators in the Kac classification is equivalent to a result of Saleur and Bauer [17] for the spin-$n$ operators in the Bethe ansatz solution of the equivalent vertex model. These are the 'boundary multi-hull' operators.

After this work was completed, we saw the paper of Shchur and Kosyakov [18], which reports Monte Carlo measurements of $P(n,1,L)$ for $n=2$ and $n=3$ on lattices with $L$ up to 64. Their quoted results agree well with our predictions in (2, 3), even though the value of $k=1$ is not large. In particular, the ratios of $-\ln P(n,1,L)$ between the cases of open boundaries (2) and periodic boundary conditions (3) is predicted to be $\frac{4}{5}=0.8$ for $n=2$ and $\frac{6}{7}\approx 0.857$ for $n=3$. The corresponding values quoted in [18] are 0.808(10) and 0.851(20). This close agreement with the asymptotic form may be explained by the observation that the higher eigenstates of $\hat{H}$ in (5) give corrections of order $\mathrm{e}^{-2\pi k}$. For $n=1$, using (7) we find a ratio $\frac{8}{5}=1.6$, to be compared with the value 1.5348 extracted from the result $P(1,1,L)\approx 0.636\,65(8)$ of Hovi and Aharony [19] for the cylinder, and the exact result of 0.5 for the square.

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